Public Key Encryption

CS 276: Introduction to Cryptography

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March 4, 2026

Overview

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2. Trapdoor Functions
3. PKE from Trapdoor Functions
4. PKE from CDH
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Public Key Encryption: Motivation

Goal

Two parties communicate with **privacy** against an eavesdropper who sees all communication. Only the holder of the secret key can decrypt.

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Why not one-way functions?

Digital signatures and symmetric-key encryption can be built from OWFs. **Lower bounds** suggest PKE *cannot* be built from OWFs alone. We need **stronger assumptions** (e.g. trapdoor functions, CDH) that imply OWFs.

PKE Scheme and Correctness

Definition 1 (Public key encryption)

A **public key encryption scheme** is a tuple $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$:

- $\mathsf{Gen}(1^n) \to (\mathsf{pk}, \mathsf{sk})$: outputs public key and secret key.
- $\mathsf{Enc}(\mathsf{pk}, m) \to c$: (probabilistic) encryption of m.
- $\mathsf{Dec}(\mathsf{sk}, c) \to m$ or $\perp$: decryption.

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Definition 2 (Correctness)

For all n, $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^n)$, $m \in \{0,1\}^*$:

$$\Pr[\mathsf{Dec}(\mathsf{sk}, \mathsf{Enc}(\mathsf{pk}, m)) = m] = 1$$

(Can be relaxed to negligible decryption error.)

IND-CPA Security

Game $\mathrm{Exp}^{\mathrm{CPA}}_{\mathcal{A}}(n)$

1. $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathrm{Gen}(1^n)$; $b \leftarrow \{0,1\}$.
2. $\mathcal{A}(\mathsf{pk})$ outputs (m_0, m_1, st).
3. $c^* \leftarrow \mathrm{Enc}(\mathsf{pk}, m_b)$.
4. $\mathcal{A}(c^*, \mathsf{st})$ outputs b'.
5. **Output** 1 if $b = b'$, else 0.

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Definition 3 (IND-CPA)

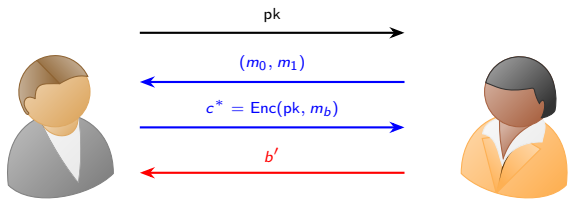
The scheme is **IND-CPA secure** if for all non-uniform PPT $\mathcal{A}$:

$$\left| \Pr[\mathsf{Exp}^{\mathsf{CPA}}_{\mathcal{A}}(n) = 1] - \frac{1}{2} \right| = \mathsf{negl}(n).$$

IND-CPA: Challenger vs Adversary

$$\mathrm{Exp}_{\mathcal{A}}^{\mathrm{CPA}}(n)$$

sample $(\mathsf{pk}, \mathsf{sk})$, $b \leftarrow \{0,1\}$



output 1 if $b' = b$; else 0

IND-CCA Security

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Same as CPA, but $\mathcal{A}$ gets a **decryption oracle** $\mathrm{Dec}(\mathrm{sk}, \cdot)$ in both phases. After receiving c^*, the oracle returns $\perp$ when queried on c^*.

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Definition 4 (IND-CCA)

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Stronger than CPA; needed for secure messaging (e.g. non-malleability).

Trapdoor Functions

Definition 5 (Trapdoor function)

A **trapdoor function** is a tuple $(\mathrm{Gen}, D, f, f^{-1})$:

- $\mathrm{Gen}(1^n) \to (s, t)$: index s, trapdoor t.
- $D(s)$: description of domain.
- $f(s, x) \to y$: evaluate on $x \in D(s)$.
- $f^{-1}(t, y) \to x$: invert using trapdoor.

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Properties

- **Correctness**: $f^{-1}(t, f(s, x)) = x$ (w.o.p.).
- **One-wayness**: Given (s, y) with $y = f(s, x)$ for random x, no PPT can find x with non-negligible probability.

PKE from TDF: Construction

Construction

Let $(\mathsf{TDF.Gen}, D, f, f^{-1})$ be a TDF. Assume a **hard-core predicate** $\mathsf{HC}(x, r)$ (e.g. from OWP).

- $\mathsf{Gen}(1^n)$: $(s, t) \leftarrow \mathsf{TDF.Gen}(1^n)$; $\mathsf{pk} = s$, $\mathsf{sk} = t$.
- $\mathsf{Enc}(\mathsf{pk}, m)$: $x \leftarrow D(s)$, $y \leftarrow f(s, x)$, $r \leftarrow \{0,1\}^{|x|}$; output $c = (y, r, m \oplus \mathsf{HC}(x, r))$.
- $\mathsf{Dec}(\mathsf{sk}, c)$: parse $c = (y, r, z)$; $x \leftarrow f^{-1}(t, y)$; output $m = z \oplus \mathsf{HC}(x, r)$.

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Proof idea

Message space $\{0,1\}$. If $\mathcal{A}$ distinguishes encryptions of 0 vs 1, we can use it to distinguish $\mathsf{HC}(x, r)$ from random and thus invert the TDF (via hard-core theorem).

PKE from CDH

Setting

$\mathbb{G}$ prime order p, generator g. CDH: given (g, g^a, g^b), computing g^{ab} is hard.

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Construction

- $\mathsf{Gen}(1^n)$: $\mathsf{sk} \leftarrow \mathbb{Z}_p$, $\mathsf{pk} \leftarrow g^{\mathsf{sk}}$; output $(\mathsf{pk}, \mathsf{sk})$.
- $\mathsf{Enc}(\mathsf{pk}, m)$: $\alpha \leftarrow \mathbb{Z}_p$, $y \leftarrow g^{\alpha}$, $k \leftarrow \mathsf{pk}^{\alpha}$; $r \leftarrow \{0,1\}^{|B|}$; output $c = (y, r, m \oplus \mathsf{HC}(k, r))$.
- $\mathsf{Dec}(\mathsf{sk}, c)$: parse $c = (y, r, z)$; $k \leftarrow y^{\mathsf{sk}}$; output $m = z \oplus \mathsf{HC}(k, r)$.

Proof: same idea as TDF-based; reduce to CDH (distinguishing g^{ab} from random via HC).

Efficiency Issue

Problem

TDF/CDH construction encrypts **one bit**; ciphertext and cost are $O(\text{poly}(n))$ per bit. For long messages: $O(M \cdot \text{poly}(n))$.

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Idea

Replace hard-core bit with a **random oracle** H to derive a **symmetric key** $k = H(x)$. Encrypt long m with a symmetric scheme under k. Ciphertext (y, z) with $z = \text{Sym.Enc}(H(x), m)$; one TDF evaluation per message.

PKE from TDF in ROM: Construction

Construction

- $\mathsf{Gen}(1^n)$: $(s, t) \leftarrow \mathsf{TDF.Gen}(1^n)$; $\mathsf{pk} = s$, $\mathsf{sk} = t$.
- $\mathsf{Enc}(\mathsf{pk}, m)$: $x \leftarrow D(s)$, $y \leftarrow f(s, x)$; $z \leftarrow \mathsf{Sym.Enc}(H(x), m)$; output $c = (y, z)$.
- $\mathsf{Dec}(\mathsf{sk}, c)$: parse $c = (y, z)$; $x \leftarrow f^{-1}(t, y)$; output $m \leftarrow \mathsf{Sym.Dec}(H(x), z)$.

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- $\mathsf{Dec}(\mathsf{sk}, c)$: parse $c = (y, z)$; $x \leftarrow f^{-1}(t, y)$; output $m \leftarrow \mathsf{Sym.Dec}(H(x), z)$.

Theorem 6

This scheme is IND-CPA secure in the **random oracle model** *assuming the TDF is one-way and the symmetric scheme is secure.*

Reduction (TDF + ROM): Sketch

$\mathcal{B}$ gets (s, y^*) (TDF challenge); goal: compute $x^* = f^{-1}(t, y^*)$.

1. Give $\mathsf{pk} = s$ to $\mathcal{A}$. Simulate H: for query q, if $f(s, q) = y^*$ then set $H(q) = k^*$ (random key used in challenge) and later output q as preimage. Else sample $H(q)$ at random and store.
2. $\mathcal{A}$ outputs (m_0, m_1). $\mathcal{B}$ picks $b \leftarrow \{0,1\}$, $k^* \leftarrow \{0,1\}^\ell$, $z^* \leftarrow \mathsf{Sym.Enc}(k^*, m_b)$, gives $c^* = (y^*, z^*)$.
3. If $\mathcal{A}$ never queries H on x^* with $f(s, x^*) = y^*$, then it never sees k^* but must distinguish $\mathsf{Sym.Enc}(k^*, m_0)$ from $\mathsf{Sym.Enc}(k^*, m_1)$ $\Rightarrow$ breaks symmetric security. So w.n.p. $\mathcal{A}$ queries $H(x^*)$; $\mathcal{B}$ outputs x^*. $\square$

Fujisaki-Okamoto and CCA

Goal

Convert **CPA-secure** (or one-way) PKE + one-time symmetric encryption into **IND-CCA secure** hybrid PKE in the **random oracle model** (Fujisaki–Okamoto, 1999; refined 2013).

Idea (high level)

- **Encrypt**: sample random σ; $K = G(\sigma)$, $c \leftarrow \mathrm{Sym.Enc}(K, m)$, $r = H(\sigma, c)$, $e \leftarrow \mathrm{Enc}(\mathrm{pk}, \sigma; r)$. Output (e, c). Two random oracles G, H; r binds e to (σ, c).
- **Decrypt**: recover σ' from e; recompute $K' = G(\sigma')$, $r' = H(\sigma', c)$; check **re-encryption** $e = \mathrm{Enc}(\mathrm{pk}, \sigma'; r')$; if yes output $\mathrm{Sym.Dec}(K', c)$, else $\perp$.

Any tampering with (e, c) fails the re-encryption check. Kyber and other NIST PKE use FO-style variants (e.g. HHK17, QROM-safe).